CSx25: Digital Signal Processing
NCS224: Signals and Systems

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Outline

- ~~Digital Signal Processing Introduction~~
  - ~~Mathematical modeling~~
  - ~~Continuous Time Signals~~
  - ~~Discrete Time Signals~~
- ~~Analyzing Continuous-Time Systems in the Time Domain~~
- ~~Analyzing Discrete Systems in the Time Domain~~
- ~~Fourier Analysis for Continuous-Time Signals and Systems~~
- **Sampling and Reconstruction**

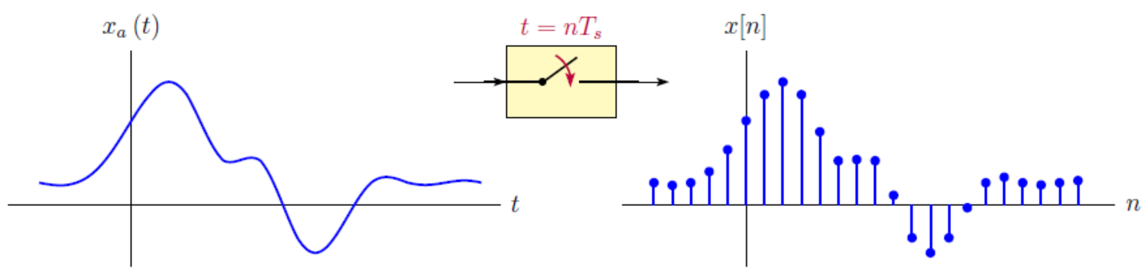
Sampling

- Sampling forms the basis of digital signals we encounter everyday in our lives.
- For example, an audio signal played back from a compact disc is a signal that has been captured and recorded at discrete time instants. When we look at the amplitude values stored on the disc, we only see values taken at equally spaced time instants (at a rate of 44,100 times per second) with missing amplitude values between these instants. This is perfectly fine since all the information contained in the original audio signal in the studio can be accounted for in these samples.
- An image captured by a digital camera is stored in the form of a dense rectangular grid of colored dots (known as pixels). When printed and viewed from an appropriate distance, we cannot tell the individual pixels apart. Similarly, a movie stored on a video cassette or a disc is stored in the form of consecutive snapshots, taken at equal time intervals. If enough snapshots are taken from the scene and are played back in sequence with the right timing, we perceive motion.

Sampling

Sampling concept

Sampling: Periodically measuring the amplitude of a continuous-time signal and constructing a discrete-time signal with the measurements.



$$x[n] = x_a(t)\Big|_{t=nT_s} = x_a(nT_s)$$

$$n:\ \text{Integer},\quad T_s:\ \text{Sampling interval/period}$$

$$f_s = \frac{1}{T_s}:\ \text{Sampling rate/frequency}$$

Sampling

Sampling the room temperature

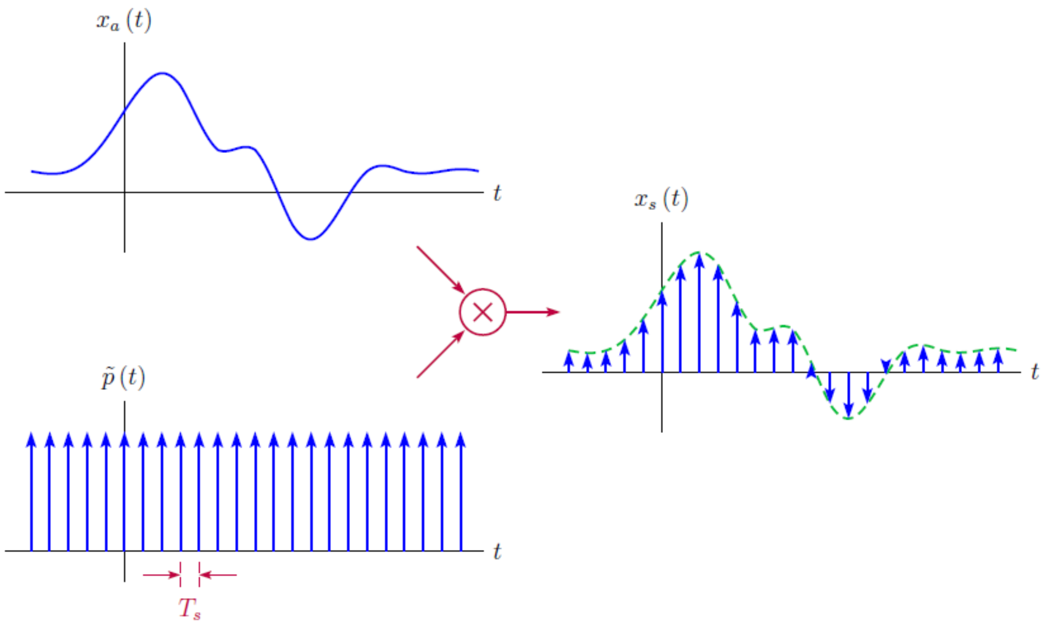
Time	8:30	8:40	8:50	9:00	9:10	9:20	9:30	9.40
Temp. (°C)	22.4	22.5	22.8	21.6	21.7	21.7	21.9	22.2
Index n	0	1	2	3	4	5	6	7

$$x[n] = \{ \underset{\substack{\uparrow \\ n=0}}{22.4}, 22.5, 22.8, 21.6, 21.7, 21.7, 21.9, 22.2, \ldots \}$$

- Do measurements taken 10 minutes apart provide enough information about the variations in temperature?
- Are we confident that no significant temperature variations occur between consecutive measurements?
- If yes, then could we have waited for 15 minutes between measurements instead of 10 minutes?

Sampling

Impulse sampling

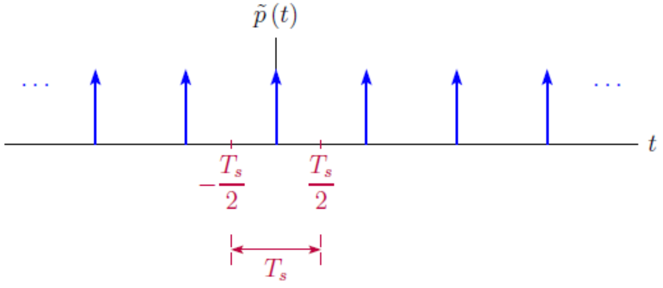


$$\tilde{p}(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_s)$$

$$x_s(t) = x_a(t)\,\tilde{p}(t) = x_a(t) \sum_{n=-\infty}^{\infty} \delta(t - nT_s) = \sum_{n=-\infty}^{\infty} x_a(nT_s)\,\delta(t - nT_s)$$

Sampling

Periodic impulse train



$$\tilde{p}(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_s)$$

EFS representation:

$$\tilde{p}(t) = \sum_{k=-\infty}^{\infty} c_k \, e^{jk\omega_s t}$$

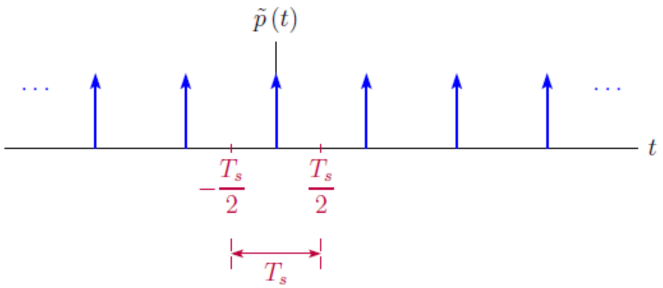
Coefficients:

$$c_k = \frac{1}{T_s} \quad \text{all } t$$

$$\begin{aligned} c_k =& \frac{1}{T_s} \int_{-T_s/2}^{T_s/2} \tilde{p}(t) \; e^{-jk\omega_s t} \, dt \\ =& \frac{1}{T_s} \int_{-T_s/2}^{T_s/2} \delta(t) \; e^{-jk\omega_s t} \, dt = \frac{1}{T_s} \; , \end{aligned}$$

Sampling

Periodic impulse train



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EFS representation:

$$\tilde{p}(t) = \sum_{k=-\infty}^{\infty} c_k \, e^{jk\omega_s t}$$

Coefficients:

$$c_k = \frac{1}{T_s} \quad \text{all } t$$

$$X_s(\omega) = \frac{1}{T_s} \sum_{k=-\infty}^{\infty} X_a(\omega - k\omega_s)$$

$$X_s(f) = \frac{1}{T_s} \sum_{k=-\infty}^{\infty} X_a(f - kf_s)$$

Sampling

The spectrum of the impulse-sampled signal is obtained by adding frequency-shifted versions of the spectrum of the original signal and then scaling the sum by 1/Ts. The terms of the summation in Eqn. (6.12) are shifted by all integer multiples of the sampling rate ωs.

The Fourier transform of the impulse-sampled signal is related to the Fourier transform of the original signal by

$$X_s(\omega) = \frac{1}{T_s} \sum_{k=-\infty}^{\infty} X_a(\omega - k\omega_s) \tag{6.12}$$

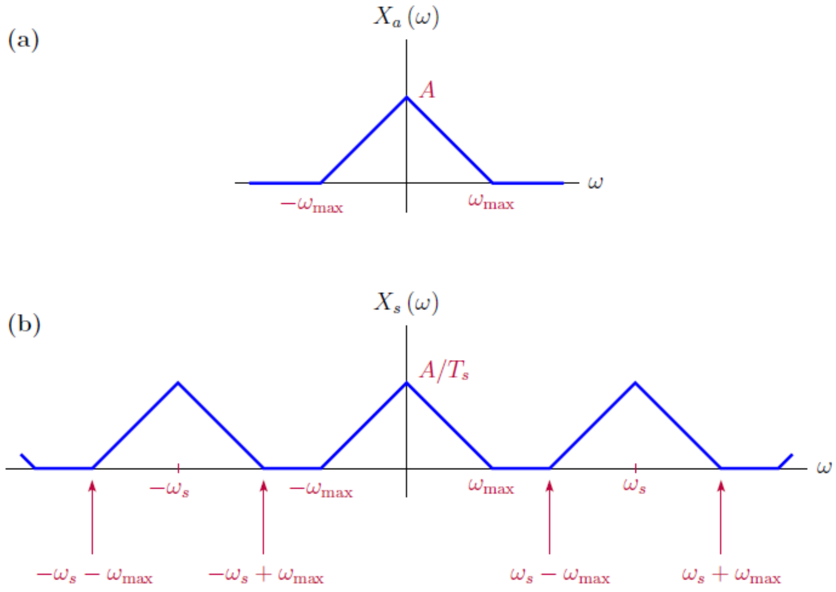
This relationship can also be written using frequencies in Hertz as

$$X_s(f) = \frac{1}{T_s} \sum_{k=-\infty}^{\infty} X_a(f - kf_s) \tag{6.13}$$

Sampling

Spectral relationships in impulse sampling (Case 1)

$$x_s(t) = \sum_{n=-\infty}^{\infty} x_a(nT_s)\,\delta(t - nT_s) \quad \Rightarrow \quad X_s(\omega) = \frac{1}{T_s} \sum_{k=-\infty}^{\infty} X_a(\omega - k\omega_s)$$

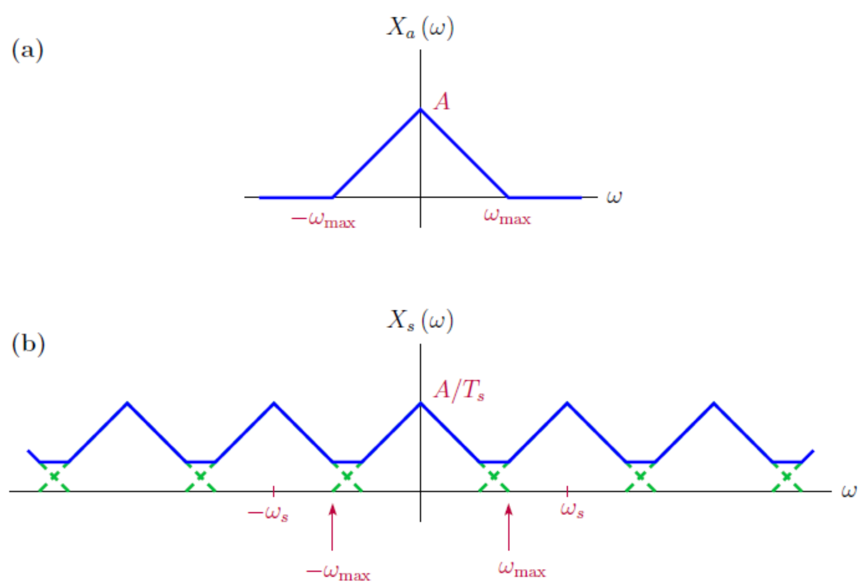


$$\omega_s \geq 2\,\omega_{\max}$$

Sampling

Spectral relationships in impulse sampling (Case 2)

$$x_s(t) = \sum_{n=-\infty}^{\infty} x_a(nT_s)\,\delta(t - nT_s) \qquad \Rightarrow \qquad X_s(\omega) = \frac{1}{T_s}\sum_{k=-\infty}^{\infty} X_a(\omega - k\omega_s)$$



$\omega_s < 2\,\omega_{\max}$

Sampling

Importance of ω_s

- If $\omega_s < 2\,\omega_{\max}$, sections of the spectrum $X_s(\omega)$ overlap with each other.
- If $\omega_s \geq 2\,\omega_{\max}$, sections of $X_s(\omega)$ do not overlap.

Conclusion

For the signal $x_a(t)$ to be recoverable from its impulse sampled version $x_s(t)$ we need

$$\omega_s \geq 2\,\omega_{\max}$$

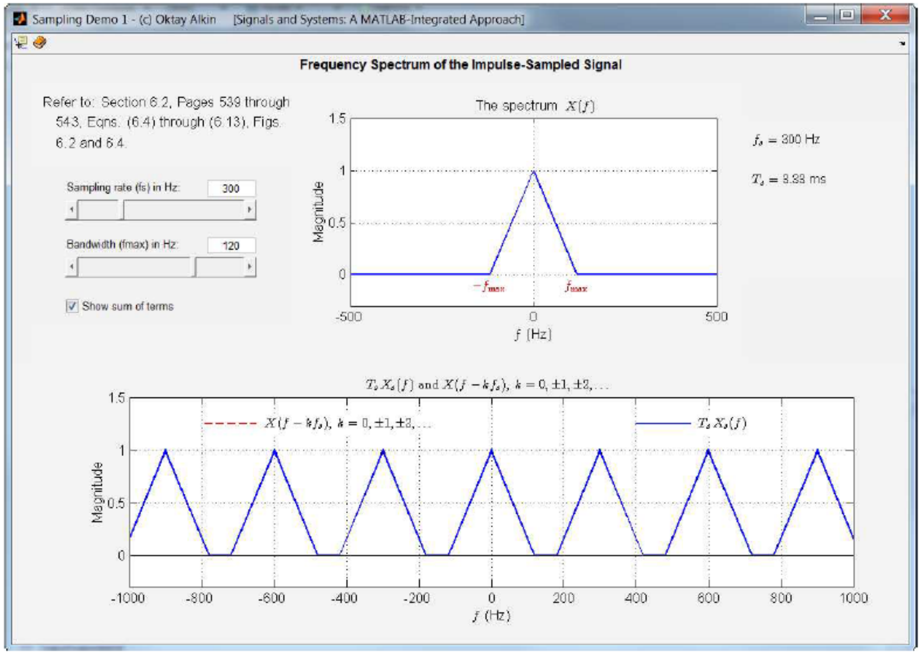
or, equivalently

$$f_s \geq 2\,f_{\max}$$

Sampling

Interactive demo: smp_demo1.m

Experiment by changing the sampling rate f_s and the bandwidth $f_{\max}$.



Sampling

Example 6.1

Impulse-sampling a right-sided exponential

Consider a right-sided exponential signal

$$x_a(t) = e^{-100t}\, u(t)$$

This signal is to be impulse sampled. Determine and graph the spectrum of the impulse sampled signal $x_s(t)$ for sampling rates $f_s = 200$ Hz, $f_s = 400$ Hz and $f_s = 600$ Hz.

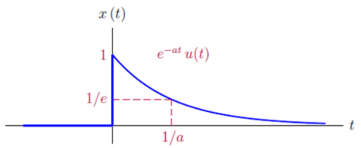
Example 4.15

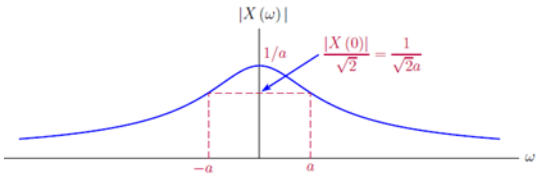
Fourier transform of a right-sided exponential signal

Determine the Fourier transform of the right-sided exponential signal

$$x(t) = e^{-at}\, u(t)$$

with $a > 0$.





Solution:

$$X(\omega) = \int_0^\infty e^{-at}\, e^{-j\omega t}\, dt = \int_0^\infty e^{-(a+j\omega)t}\, dt = \frac{1}{a+j\omega}$$

Sampling

Example 6.1

Impulse-sampling a right-sided exponential

Consider a right-sided exponential signal

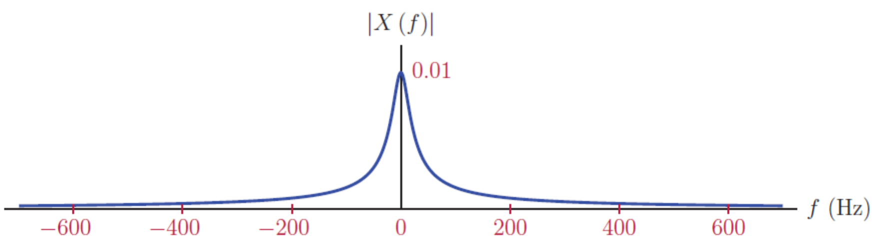
$$x_a(t) = e^{-100t}\, u(t)$$

This signal is to be impulse sampled. Determine and graph the spectrum of the impulse sampled signal $x_s(t)$ for sampling rates $f_s = 200$ Hz, $f_s = 400$ Hz and $f_s = 600$ Hz.

Solution: Using the techniques developed in Chapter 4, the frequency spectrum of the signal $x_a(t)$ is

$$X_a(f) = \frac{1}{100 + j2\pi f}$$

which is graphed in Fig. 6.6(a).



Sampling

Example 6.1

Impulse-sampling a right-sided exponential

Consider a right-sided exponential signal

$$x_a(t) = e^{-100t}\, u(t)$$

This signal is to be impulse sampled. Determine and graph the spectrum of the impulse sampled signal $x_s(t)$ for sampling rates $f_s = 200$ Hz, $f_s = 400$ Hz and $f_s = 600$ Hz.

Solution:

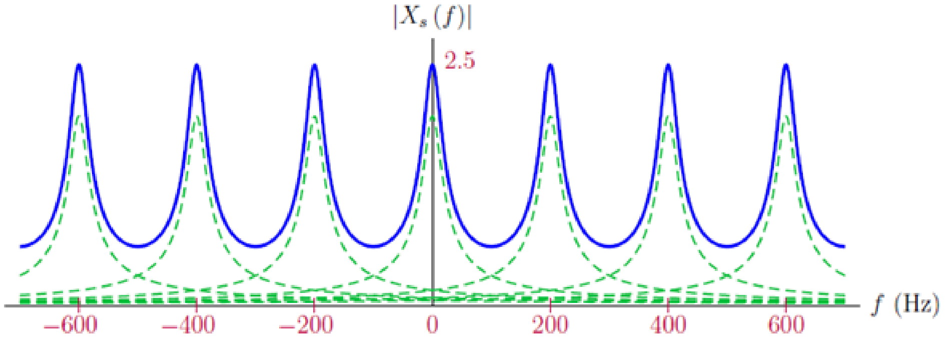
$$X_a(f) = \frac{1}{100 + j2\pi f}$$

$$x_s(t) = \sum_{n=0}^{\infty} e^{-100nT_s}\, \delta(t - nT_s) \qquad \Rightarrow \qquad X_s(f) = \frac{1}{1 - e^{-100T_s}\, e^{-j2\pi f T_s}}$$

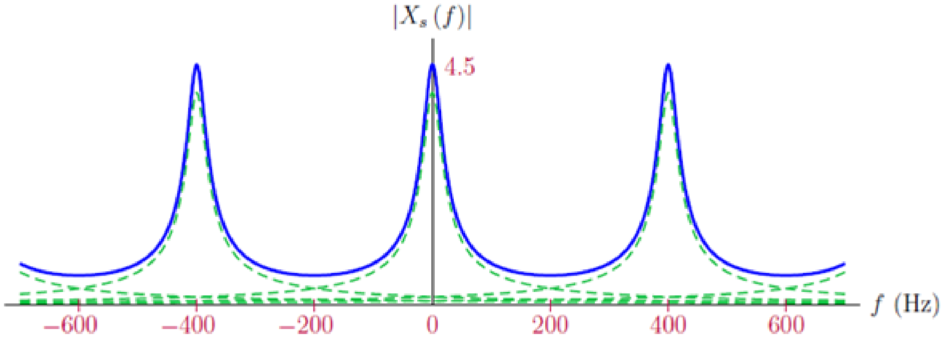
Sampling

Example 6.1. (continued)

$f_s = 200 \;\; \mathrm{Hz}$



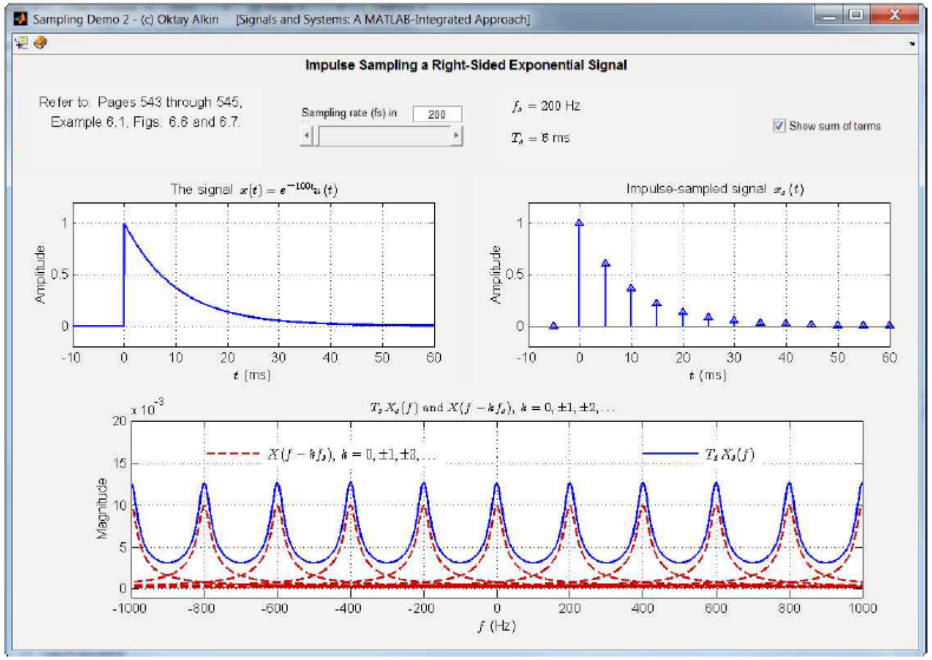
$f_s = 400 \;\; \mathrm{Hz}$



Sampling

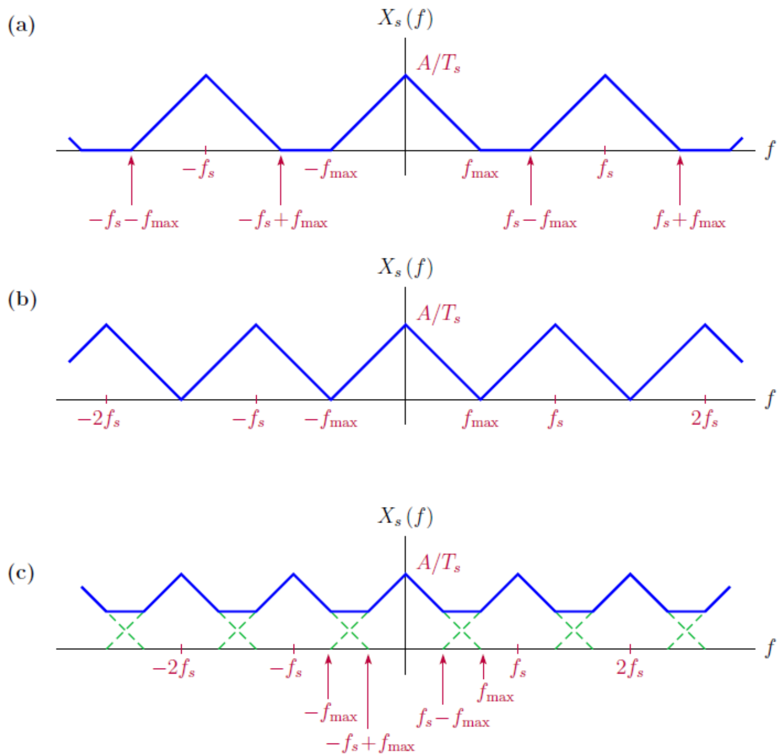
Interactive demo: smp_demo2.m

Experiment by changing the sampling rate f_s.



Sampling

Nyquist sampling criterion



$f_s > 2\,f_{\max}$

$f_s = 2\,f_{\max}$

$f_s < 2\,f_{\max}$

Sampling

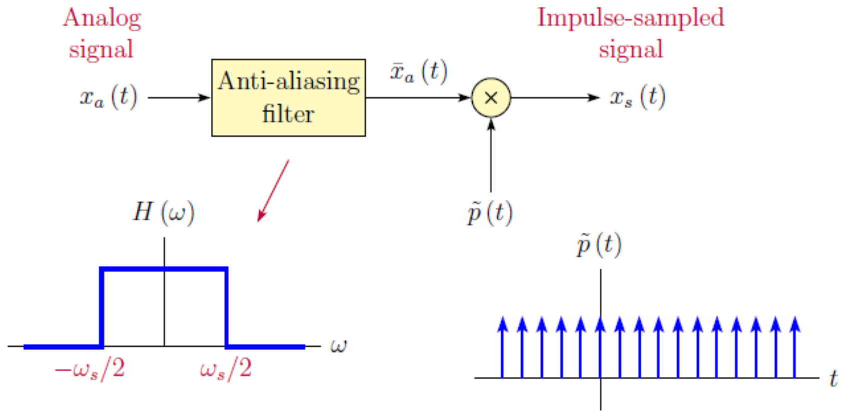
For the impulse-sampled signal to form an accurate representation of the original signal, the *sampling rate must be at least twice the highest frequency* in the spectrum of the original signal. This is known as the **Nyquist** sampling criterion. It was named after Harry Nyquist (1889-1976) who first introduced the idea in his work on telegraph transmission. Later it was formally proven by his colleague Claude **Shannon** (1916-2001) in his work that formed the foundations of information theory.

The signal is processed through an *anti-aliasing filter* before it is sampled, effectively removing all frequencies that are greater than half the sampling rate.

Sampling

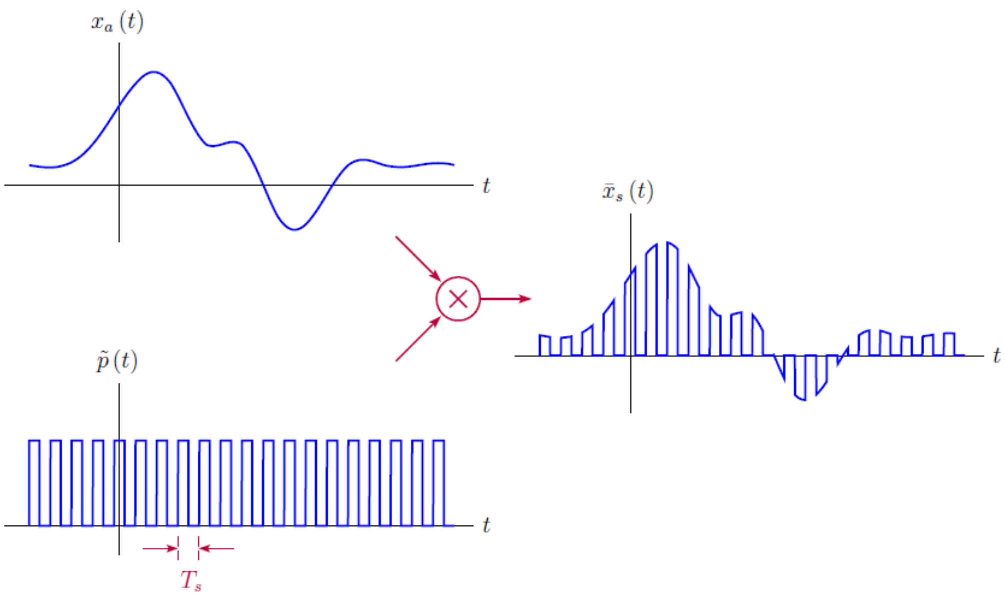
Use of anti-aliasing filter

- In practical implementations, the sampling rate f_s is fixed by hardware constraints.
- The highest frequency $f_{\max}$ is not always known a priori.
- An *anti-aliasing filter* is used to avoid aliasing.



Sampling

Natural sampling

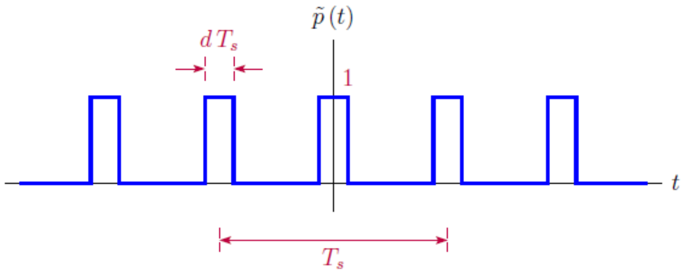


$$\bar{x}_s(t) = x_a(t)\,\tilde{p}(t)$$
$$= x_a(t) \sum_{n=-\infty}^{\infty} \Pi\left(\frac{t-nT_s}{dT_s}\right)$$

d : Duty cycle

Sampling

Periodic pulse train



$$\tilde{p}(t) = \sum_{n=-\infty}^{\infty} \Pi\left(\frac{t - nT_s}{dT_s}\right)$$

EFS representation:

$$\tilde{p}(t) = \sum_{k=-\infty}^{\infty} c_k\, e^{jk\omega_s t}$$

Coefficients:

$$c_k = d\,\mathrm{sinc}(kd)$$

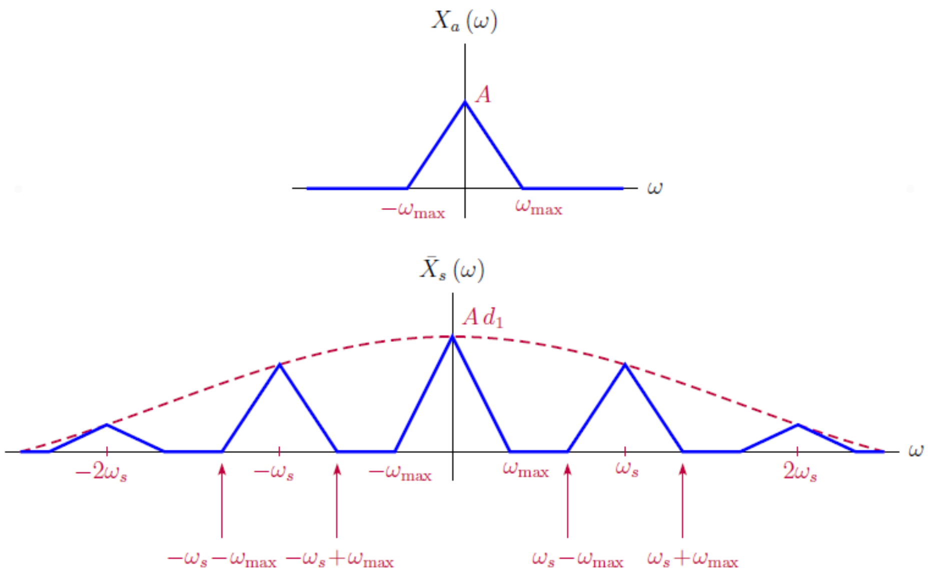
Spectrum of naturally sampled signal

$$\bar{X}_s(\omega) = d \sum_{k=-\infty}^{\infty} \mathrm{sinc}(kd)\; X_a(\omega - k\omega_s)$$

Sampling

Spectral relationships in natural sampling (Case 1)

$$\bar{X}_s(\omega) = d \sum_{k=-\infty}^{\infty} \operatorname{sinc}(kd)\, X_a(\omega - k\omega_s)$$



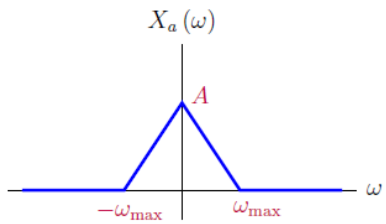
Duty cycle:

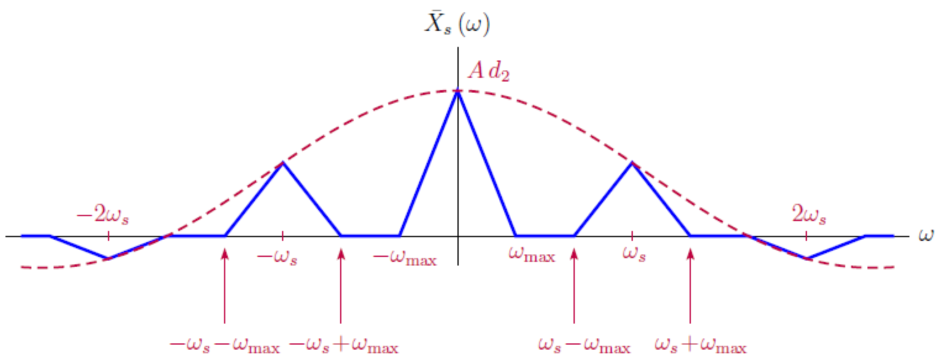
$d = d_1$

Sampling

Spectral relationships in natural sampling (Case 2)

$$\bar{X}_s(\omega) = d \sum_{k=-\infty}^{\infty} \operatorname{sinc}(kd)\, X_a(\omega - k\omega_s)$$





▸ MATLAB Exercise 6.4

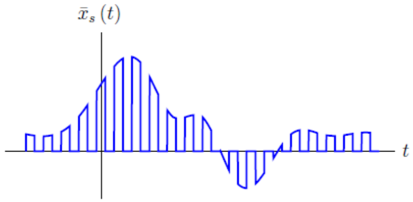
▸ MATLAB Exercise 6.6

Duty cycle:

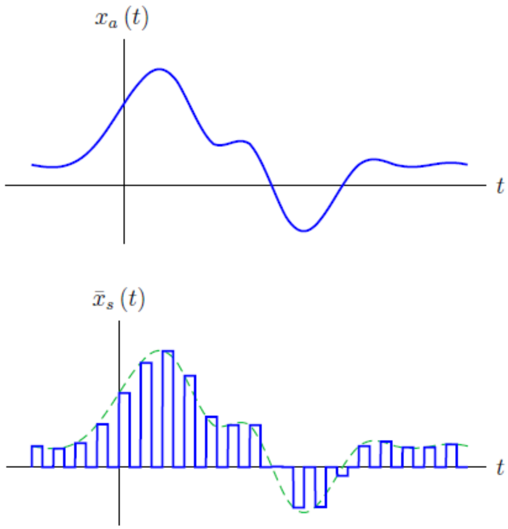
$d = d_2 > d_1$

Sampling

Zero-order hold sampling



Sample $x_a(t)$ at T_s second intervals, and hold the amplitude constant for the duration of the pulse (dT_s seconds).



In natural sampling the tops of the pulses are not flat but are rather shaped by the signal xa (t). This behavior is not always desired, especially when the sampling operation is to be followed by **conversion of each pulse to digital format**. An alternative is to hold the amplitude of each pulse constant, equal to the value of the signal at the left edge of the pulse. This is referred to as zero-order hold sampling.

Reconstruction

Often the purpose of sampling an analog signal is to store, process and/or transmit it digitally, and to later *convert it back to analog format*.

To that end, one question still remains: How can the original analog signal be reconstructed from its sampled version? Given the discrete-time signal x[n] or the impulse-sampled signal xs (t), how can we obtain a signal identical, or at least reasonably similar, to xa (t)? Obviously we need a way to *"fill the gaps"* between the impulses of the signal xs (t) in some meaningful way.

In more technical terms, signal amplitudes between sampling instants need to be computed by some form of **interpolation**.

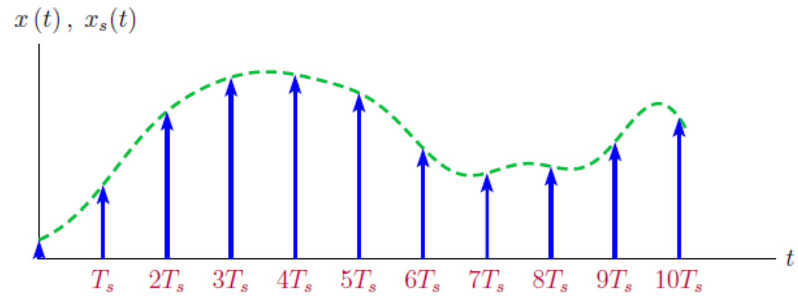
Reconstruction

Reconstruction of a signal from its sampled version

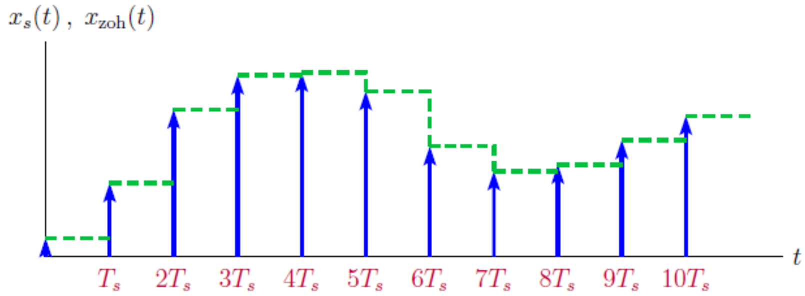
Impulse sampling of a signal $x_a(t)$:

$$x_s(t) = \sum_{n=-\infty}^{\infty} x_a(nT_s)\,\delta(t - nT_s)$$

Impulse sampling



Reconstruction using zero-order hold

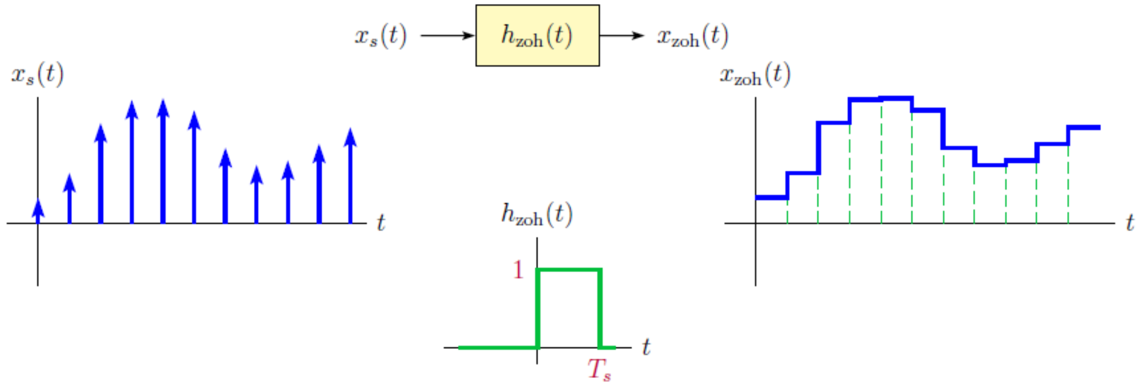


Reconstruction

Reconstruction of a signal from its sampled version (continued)

Zero-order-hold interpolation filter:

$$h_{\mathrm{zoh}}(t) = \Pi\left(\frac{t - T_s/2}{T_s}\right)$$

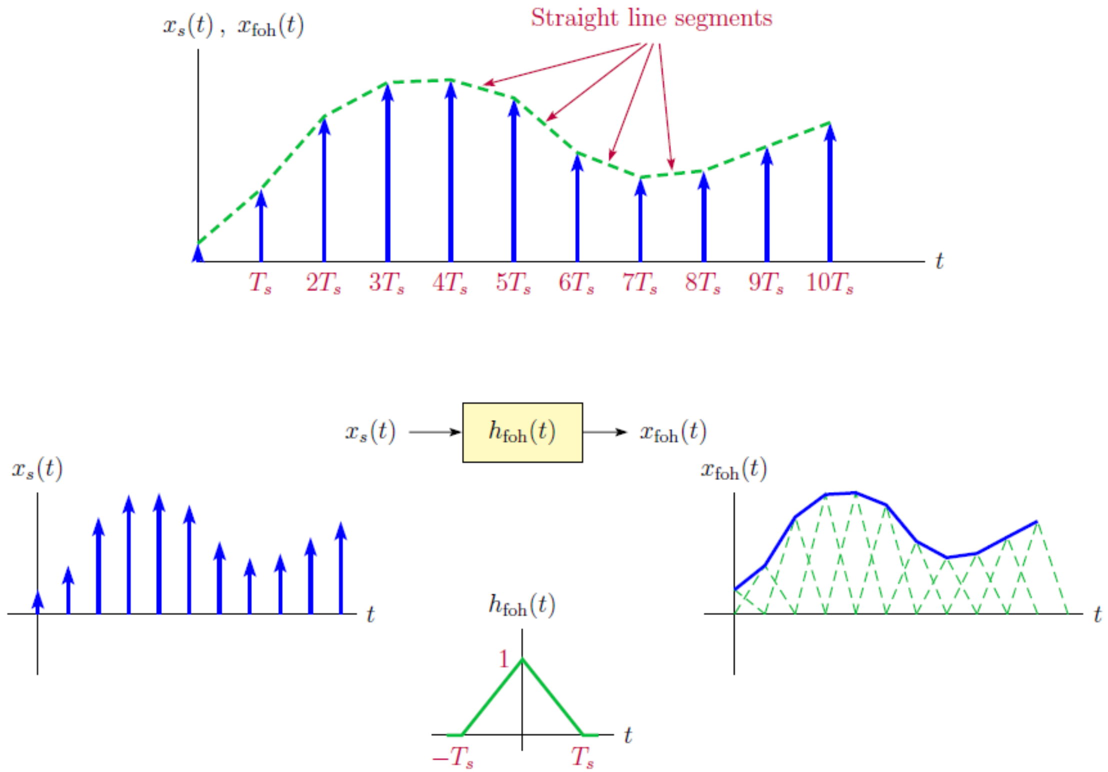


$$X_{zoh}(\omega) = \mathrm{sinc}\left(\frac{\omega T_s}{2\pi}\right) e^{-j\omega T_s/2} \sum_{k=-\infty}^{\infty} X_a(\omega - k\omega_s)$$

Reconstruction

Reconstruction of a signal from its sampled version (continued)

First-order hold interpolation:



Reconstruction

Reconstruction of a signal from its sampled version (continued)

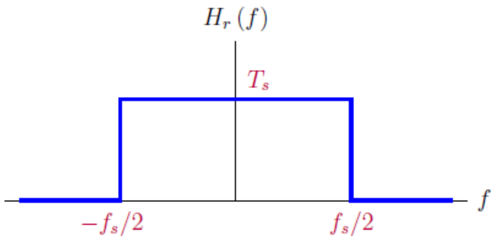
Ideal reconstruction

Spectrum of impulse sampled signal:

$$X_s(f) = \frac{1}{T_s} \sum_{k=-\infty}^{\infty} X_a(f - kf_s)$$

Ideal lowpass reconstruction filter:

$$H_r(f) = T_s \, \Pi\left(\frac{f}{f_s}\right)$$

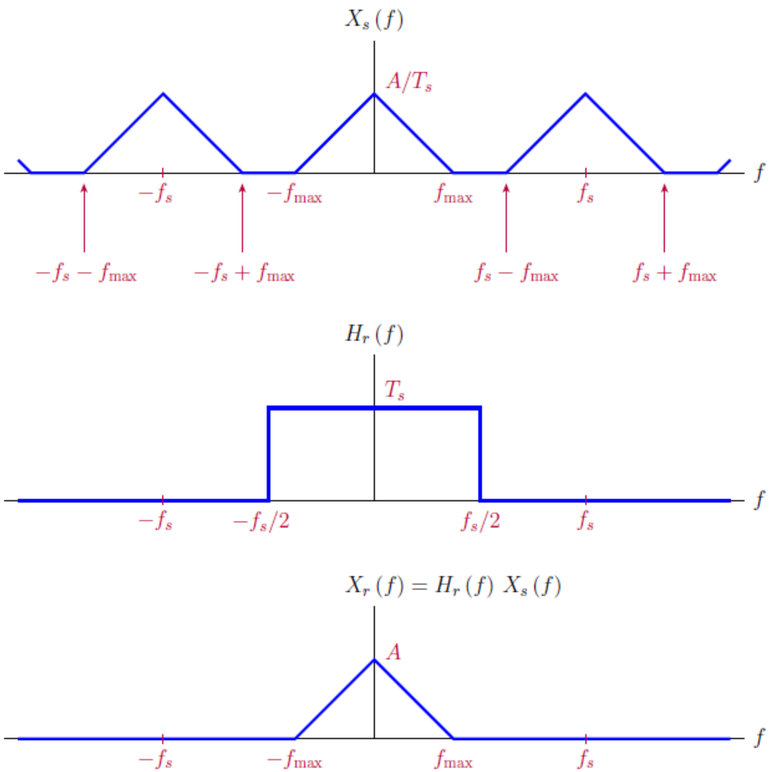


The output of the ideal lowpass filter is

$$X_r(f) = H_r(f)\; X_s(f) = T_s \, \Pi\left(\frac{f}{f_s}\right) \frac{1}{T_s} \sum_{k=-\infty}^{\infty} X_a(f - kf_s) = X_a(f)$$

Reconstruction

Reconstruction of a signal from its sampled version (continued)



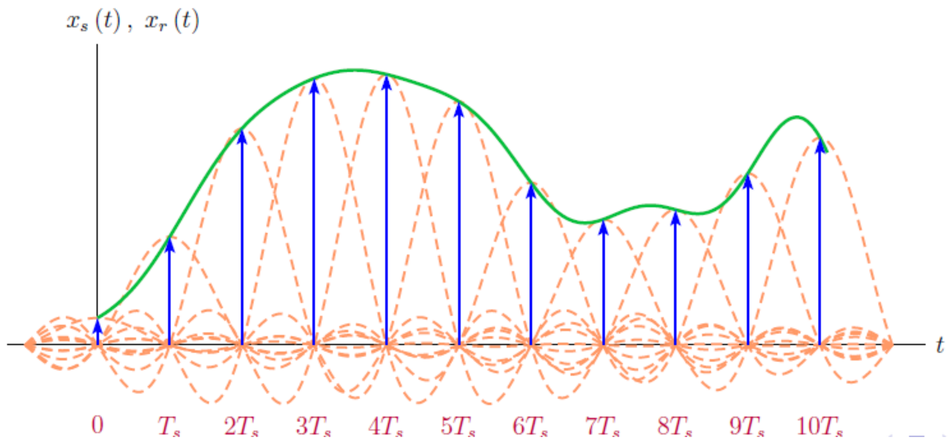
Reconstruction

Reconstruction of a signal from its sampled version (continued)

The impulse response of the ideal lowpass filter is

$$h_r(t) = \text{sinc}(t f_s) = \text{sinc}\left(\frac{t}{T_s}\right) \tag{1}$$

- The output $x_r(t)$ of the ideal lowpass reconstruction filter is equal to the sampled signal at each sampling instant.
- In between sampling instants $x_r(t)$ is obtained by interpolation through the use of sinc functions. This is referred to as *bandlimited interpolation*.



▸ MATLAB Exercise 6.7

▸ MATLAB Exercise 6.8

Resampling Discrete-Time Signals

Reducing the sampling rate by an integer factor

Reduce the sampling rate by a factor of D:

$$x_d[n] = x[nD]$$

This operation is known as *downsampling*.

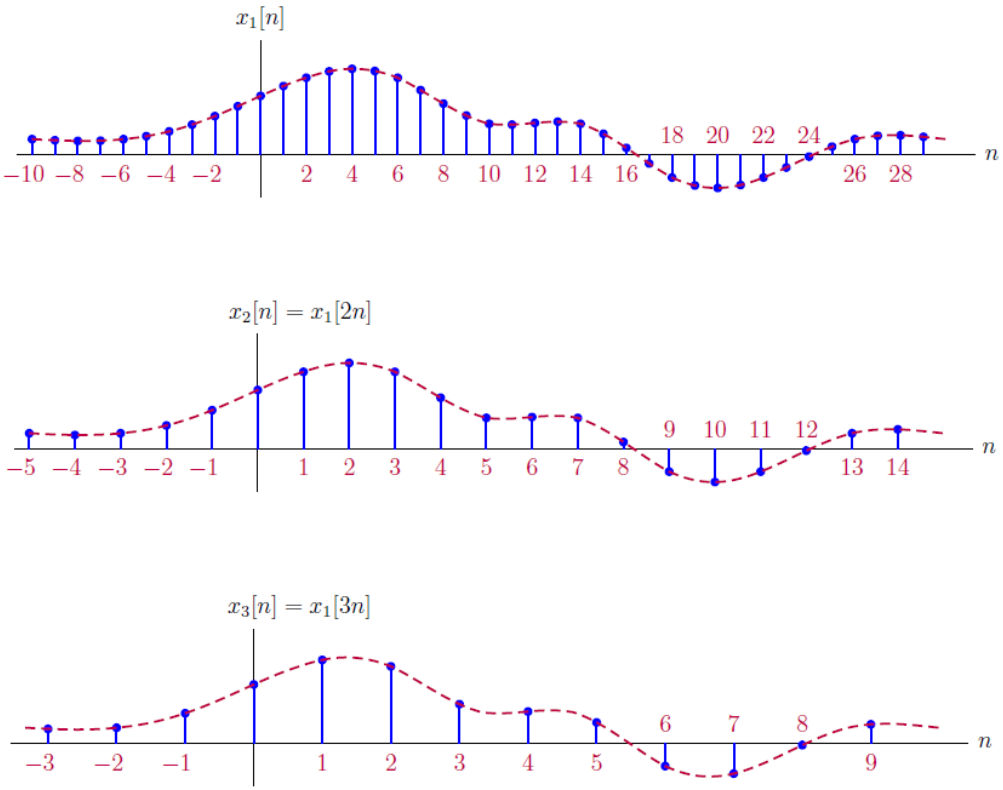
The parameter D is the *downsampling rate*.

$$x[n] \longrightarrow \boxed{\downarrow D} \longrightarrow x_d[n]$$

- The signal $x_d[n]$ retains one sample out of each set of D samples of the original signal $x[n]$.
- For each sample retained, $(D - 1)$ samples are discarded.

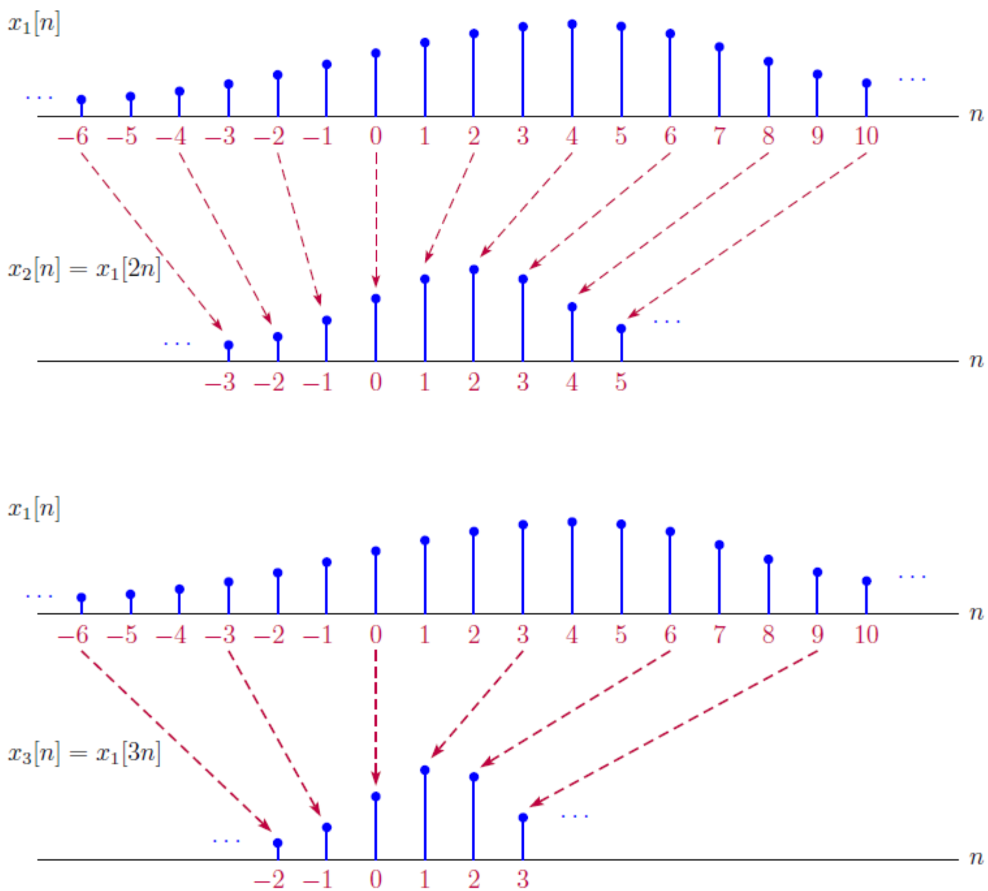
Resampling Discrete-Time Signals

Reducing the sampling rate by an integer factor (continued)



Resampling Discrete-Time Signals

Reducing the sampling rate by an integer factor (continued)



Resampling Discrete-Time Signals

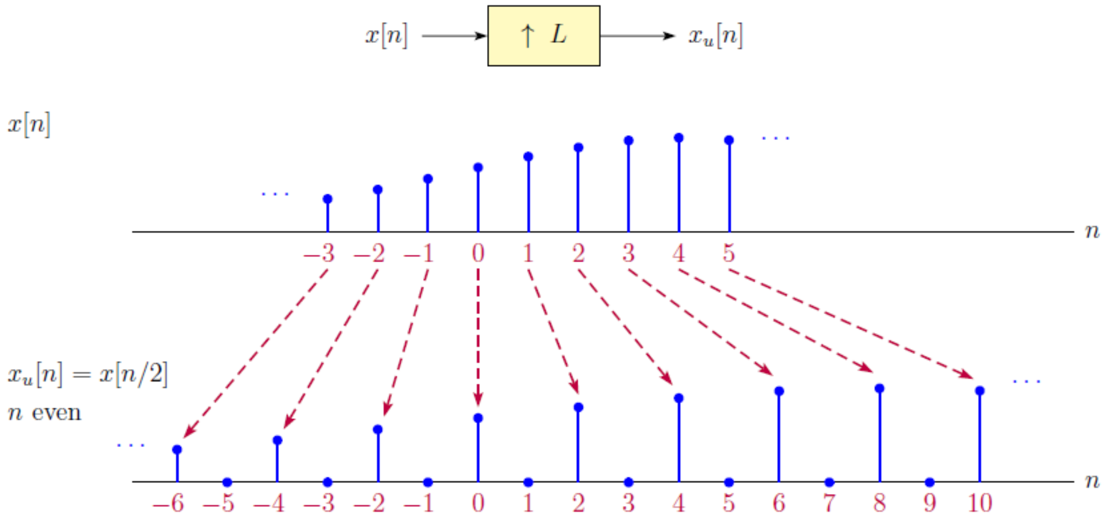
Increasing the sampling rate by an integer factor

Let

$$x_u[n] = \begin{cases} x[n/L]\,, & n = kL\,, \quad k: \text{integer} \\ 0\,, & \text{otherwise} \end{cases}$$

This operation is known as *upsampling*.

The parameter L is the *upsampling rate*.



Resampling Discrete-Time Signals

Increasing the sampling rate by an integer factor (continued)

A lowpass *interpolation filter* is needed to make the zero-amplitude samples "blend-in" with the rest of the signal.